# SMOOTH MANIFOLDS FALL 2022 - HOMEWORK 5

**Problem 1.** Show that the distribution $E(p) = \{v \in \mathbb{R}^3 : \langle v, p\rangle = 0\}$ on $\mathbb{R}^3 \setminus \{0\}$ is involutive directly. What foliation does this integrate to? [*Hints*: Involutivity is a local property. You may use different vectors fields to frame at different points. Why won't you be able to find a single pair of vector fields to frame $E$ globally? Finally, there are good and bad choices here. Part of the difficulty is finding "nice" fields. In particular, you can find linear fields.]

*Solution.* Consider the following vector fields

$$X_1 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \qquad X_2 = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x} \qquad X_3 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$$

From direct computation, it follows that $X_i$ is subordinate to $E$ for every $i$, and that $[X_i, X_j] = \pm X_k$ whenever $X_i$, $X_j$ and $X_k$ are all distinct. Furthermore, we wish to show that $\dim(\mathbb{R}X_1(p) + \mathbb{R}X_2(p) + \mathbb{R}X_3(p)) > 1$ for all $p \in \mathbb{R}^3 \setminus \{0\}$. This fails if and only if all of the $X_i$ are proportional. That is, at $p = (x, y, z)$, since $X_1$ is proprtional to $X_3$, it follows that $y = 0$. Repeating this for each other pair, we conclude that $p = 0$, a contradiction. Hence at least two such vectors are not proportional, and at each point $p$, the vectors $X_i(p)$ span $E(p)$. This implies that $E$ is involutive. □

**Problem 2.** Prove the other direction of the Frobenius theorem. That is, show that if $\mathcal{F}$ is a foliation, then $E := T\mathcal{F}$ is involutive.

*Solution.* Let $X$ and $Y$ be vector fields subordinate to $E$, and $p \in M$. Let $\varphi : U \to \mathbb{R}^k \times \mathbb{R}^{n-k}$ be a foliation chart of $\mathcal{F}$, so the local leaves of $\mathcal{F}$ are the preimages of the flats $\mathbb{R}^k \times \{y\} \subset \mathbb{R}^n$. This implies that $d\varphi(E(p)) = \mathbb{R}^k \times \{0\}$, and

$$d\varphi(X) = \sum_{i=1}^{k} a_i \frac{\partial}{\partial x_i} \qquad d\varphi(Y) = \sum_{i=1}^{k} b_i \frac{\partial}{\partial x_i}$$

for some $C^\infty$ functions $a_i$ and $b_i$, since $X$ and $Y$ are subordinate to $E$. It follows that the same is true for $[d\varphi(X), d\varphi(Y)]$, since

$$[d\varphi(X), d\varphi(Y)] = \left(\sum_{i,j} \frac{\partial b_i}{\partial x_j} - \frac{\partial a_i}{\partial x_j}\right) \frac{\partial}{\partial x_i}.$$

Hence $[X, Y]$ is subordinate to $E$, since $[X, Y] = d\varphi^{-1}[d\varphi(X), d\varphi(Y)]$. □

**Problem 3.** Let $X$ and $Y$ be vector fields on a smooth manifold $M$. Show that if $\varphi_t^X \circ \varphi_s^Y(x) = \varphi_{f(t,x)s}^Y \circ \varphi_t^X(x)$ for some smooth function $f : \mathbb{R} \times M \to \mathbb{R}$, all $s, t \in \mathbb{R}$ and $x \in M$, then $[X, Y]$ is a multiple of $Y$. Compute it in terms of the derivatives of $f$.

*Proof.* Recall that

$$[X, Y](p) = -\left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{ds}\right|_{s=0} \varphi_t^X \varphi_s^Y \varphi_{-t}^X(p).$$

Since $\varphi_t^X \varphi_s^Y \varphi_{-t}^X = \varphi_{f(t,x)s}^Y$, we must compute

$$\left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{ds}\right|_{s=0} \varphi^Y_{f(t,x)s}(p) = \left.\frac{d}{dt}\right|_{t=0} f(t,x)Y(p) = \frac{df}{dt}(0,x)Y(p).$$

So $[X,Y](p) = -\frac{df}{dt}(0,x)Y(p)$. □